PU.M.A. Vol. **23** (2012), No. 1, pp. 49–60

A special learning process with time delay

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(Received: April 19, 2012)

Abstract. A special learning process is examined in linear N-firm oligopolies in which the firms adjust their beliefs on the price function adaptively based on predicted and actual prices. It is assumed that the price information is delayed, which results in a system of difference-differential equations. In the case of identical speed of adjustment a complete spectrum analysis is given which leads to determining the stability region as well as stability switches. It is shown that only the smallest such threshold gives stability switch, where Hopf bifurcation occurs.

1 Introduction

Dynamic models play an important role in quantitative sciences including engineering, biology and social sciences. There is a large literature examining the existence and uniqueness of steady states and their asymptotic properties. In mathematical economics oligopoly models are considered as one of the most frequently studied areas of research. The classical model assumes a set of firms producing identical goods to a homogeneous market (Cournot, 1838). Their competition is modeled as a N-person noncooperative game. The existence and uniqueness of the equilibrium was the main focus first under different conditions and later the research turned to the dynamic extentions of this game. Okuguchi (1976) gives a comprehensive summary of the earlier results, and their multiproduct extensions with applications are discussed in Okuguchi and Szidarovszky (1999). Most models were linear, when local stability implied global stability. In recent years an increasing attention has been given to nonlinear models, where the global asymptotical behavior of the state trajectories has a huge variety from global asymptotical stability to chaotic behavior. Recent developments of nonlinear oligopolies are reported for example, in Bischi et al. (2010). In most models discussed in the literature it was assumed that the firms had complete knowledge of each others' technology as well as of the market. In real economies it is realistic to assume that the firms know the cost functions of the competitors, however the market demand function as well as the unit price function are always uncertain. However by repeated price observations they are able to continuously update their beliefs of the price function through a learning process, which is modeled as a dynamic system. Fudenberg and Levin (1998) give a general theory of learning in games. Marimon (1997) and Kirman and Salmon (1995) have to be also mentioned as important references. In the case of oligopolies the uncertainty of the price function and special learning processes are discussed in Bischi et al. (2010) and the references given there. In all earlier studies instantaneous price information was assumed, which is not realistic in real economies. By introducing delayed information on the market price the asymptotical properties of the resulting dynamic models become much more complicated. If continuously distributed delay is assumed then the model is a Volterra-type integro-differential equation. In the case of gamma-density-type weighting functions the spectrum is finite (Cushing, 1977), so complete eigenvalue analysis can be given in important special cases (Chiarella and Szidarovszky, 2004). By assuming fixed delays the governing dynamics is formulated as a delay differential equation, the characteristic equation of which is transcendental with an infinite spectrum. The asymptotic behavior of delay differential equations have a very large literature. Burger (1956), Cooke and Grossman (1982), Bellman and Cooke (1956) provide complete stability analysis in several important special cases of single delays. In the presence of multiple delays the analysis becomes much more complicated (Hale and Huang, 1993). In this paper a special learning process in classical Cournot oligopolies will be revisited. This model was introduced in Szidarovszky and Krawczyk (2004), and further examined in Bischi et al. (2010). Without information delay the system is always asymptotically stable showing that the beliefs of the firms about the price function converge to the true function as time goes to infinity. This stability however might be lost in the presence of fixed delay. We will present a complete stability analysis of the delay model. This paper develops as follows. The examined learning process will be introduced in Section 2, and complete stability analysis will be provided in Section 3. Section 4 discusses stability switches and the appearance of Hopf bifurcation. Conclusions, economic interpretations of the results, and further research directions will be outlined in the final section.

2 A special learning process

Consider an industry of N firms that produce the same product to a homogeneous market. Let x_k denote the output of firm k, then $s = \sum_{i=1}^{N} x_i$ is the total output of the industry and $s_k = \sum_{i \neq k} x_i$ is the output of the rest of the industry from the viewpoint of firm k. The unit price is a strictly decreasing function of the supply, $p(s) = B - As$, where B is the maximum price and $-A$ the marginal price. It is assumed that both A and B are positive. The cost function of firm k is also assumed to be linear, $C_k(x_k) = c_k x_k + d_k$, where d_k is the fixed cost and c_k is the marginal cost. It is also assumed that the technology of the competitors is known by each firm, so that the cost functions are known by each of them. However, they can have only an estimate of the price function. By assuming that the marginal price is a common knowledge, each firm has only an estimate of the maximum price. Let's examine this situation from the point of view of firm k. If its estimate of the maximum price is B_k, then it belives that the price function is $p_k(s) = B_k - As$. Therefore it also believes that the profit of any firm l, including itself is

$$\varphi_l = x_l(B_k - As_l - Ax_l) - (c_l x_l + d_l). \tag{1}$$

This is a concave parabola in x_l. Firm k believes that the profit maximizing output of firm l is positive, otherwise this firm would leave the industry with zero optimal production level. The first order condition shows that at the optimum

$$B_k - As_l - 2Ax_l - c_l = 0,$$

that is,

$$B_k - A(s - x_l) - 2Ax_l - c_l = 0,$$

which results in the believed best response of firm l:

$$x_l = \frac{B_k - As - c_l}{A}. \tag{2}$$

At the believed equilibrium every firm selects its best response, so equation (2) should hold for all firms l. By adding this equation for all firms,

$$s = \frac{1}{A}\left(NB_k - NAs - \sum_i c_i\right) \tag{3}$$

from which the believed output of the industry becomes

$$s = \frac{NB_k - \sum_i c_i}{(N+1)A}$$

with believed equilibrium price

$$p_k = B_k - As = \frac{B_k + \sum_i c_i}{N+1}. \tag{4}$$

Based on equation (2), firm k believes that its equilibrium output is

$$x_k = \frac{B_k - As - c_k}{A} = \frac{B_k + \sum_{i=1}^N c_i - (N+1)c_k}{A(N+1)}. \tag{5}$$

In reality, however, each firm has its own estimate B_l of the maximum price, so their beliefs of the equilibrium industry output as well as that of the equilibrium price are usually different. Each of them provides its own equilibrium output (2) when k is replaced by l, so in reality the industry output becomes

$$\sum_{k=1}^N x_k = \frac{1}{A(N+1)}\left(\sum_{k=1}^N B_k - \sum_{k=1}^N c_k\right) \tag{6}$$

with the actual market price

$$p = B - A\sum_{k=1}^{N} x_k = B - \frac{1}{N+1}\left(\sum_{k=1}^{N} B_k - \sum_{k=1}^{N} c_k\right). \tag{7}$$

It is assumed now that the firms at each time period $t \geq 0$ assess their beliefs of the equilibrium based on their most current estimate $B_k(t)$ of the maximum price, so firm k believes that the market price will be

$$p_k(t) = \frac{1}{N+1}\left(B_k(t) + \sum_{i=1}^{N} c_i\right), \tag{8}$$

however, its price information from the market has some delay $\tau > 0$. Therefore the price that firm k believes at time period t is really the market price of the earlier time period $t - \tau$:

$$p(t-\tau) = B - \frac{1}{N+1}\left(\sum_{l=1}^{N} B_l(t-\tau) - \sum_{l=1}^{N} c_l\right). \tag{9}$$

The presence of delay in the price information is a realistic assumption, since sales and price information from the wholesalers to the manufacturers are sent periodically and not continuously at each time period. If the believed price is lower than the actual price, then the firm wants to increase its price belief by increasing the value of B_k. If the believed price is higher than the actual price, then firm k wants to decrease its price function by decreasing B_k, and if the believed and actual prices are equal, then the firm does not have any reason to change its belief on the price function. This adjustment process can be described by the dynamic equation

$$\dot{B}_k(t) = K_k\left(p(t-\tau) - p_k(t)\right), \tag{10}$$

where $K_k > 0$ is the speed of adjustment of firm k. By using equations (8) and (9) a system of delay differential equations is obtained:

$$\dot{B}_k(t) = \frac{K_k}{N+1}\left((N+1)B - \sum_{l=1}^{N} B_l(t-\tau) - B_k(t)\right) \tag{11}$$

for $k = 1, 2, \cdots, N$. By introducing the new variables $\Delta_k = B_k(t) - B$ and $\alpha_k = \frac{K_k}{N+1}$, this system can be simplified as

$$\dot{\Delta}_k(t) + \alpha_k\Delta_k(t) + \alpha_k\sum_{i=1}^{N}\Delta_i(t-\tau) = 0 \quad (1 \leq k \leq N). \tag{12}$$

3 Stability analysis

As usual, we are looking for the solution in the form of $\Delta_k(t) = e^{\lambda t} u_k$. Substituting this solution into equation (12) we get

$$\lambda e^{\lambda t} u_k + \alpha_k e^{\lambda t} u_k + \alpha_k \left(\sum_{i=1}^{N} u_i \right) e^{\lambda(t-\tau)} = 0,$$

that is

$$\left(\lambda \underline{I} + \underline{D} + \underline{a}\, \underline{1}^T e^{-\lambda\tau} \right) \underline{u} = \underline{0}, \tag{13}$$

where $\underline{I}$ is the $N \times N$ identity matrix, $\underline{D} = \text{diag}\,(\alpha_1, \cdots, \alpha_N)$, $\underline{a} = (\alpha_1, \alpha_2, \cdots, \alpha_N)^T$, $\underline{1}^T = (1, 1, \cdots, 1)$ and $\underline{u} = (u_1, u_2, \cdots, u_N)^T$. Nontrivial solution exists if and only if the determinant of (13) is zero. In order to have a simple expression for this determinant, we use the simple fact that if $\underline{p}$ and $\underline{q}$ are N-element column vectors, then

$$\det(\underline{I} + \underline{p}\, \underline{q}^T) = 1 + \underline{q}^T\, \underline{p}\,. \tag{14}$$

A simple proof of this identity is given for example, in Bischi et al. (2010). So we have the following equation:

$$\begin{aligned} 0 &= \det(\lambda \underline{I} + \underline{D} + \underline{a}\, \underline{1}^T e^{-\lambda\tau}) \\ &= \det(\lambda \underline{I} + \underline{D}) \det(\underline{I} + (\lambda \underline{I} + \underline{D})^{-1} \underline{a}\, \underline{1}^T\, e^{-\lambda\tau}) \\ &= \prod_{k=1}^{N} (\lambda + \alpha_k) \left(1 + \sum_{k=1}^{N} \frac{\alpha_k}{\lambda + \alpha_k} e^{-\lambda\tau} \right). \end{aligned}$$

The eigenvalues are $\lambda = -\alpha_k$ $(k = 1, 2, \cdots, N)$ and the solutions of equation

$$1 + \sum_{k=1}^{N} \frac{\alpha_k}{\lambda + \alpha_k} e^{-\lambda\tau} = 0. \tag{15}$$

Since $-\alpha_k < 0$, it is sufficient to find conditions that the roots of this equation have negative real parts in order to guarantee asymptotical stability in which case the beliefs of the firms converge to the true price function as t converges to infinity. In order to simplify the mathematical analysis assume that the firms select identical speed of adjustment $K_k \equiv K$ for $k = 1, 2, \cdots, N$, so $\alpha_k \equiv \gamma$. Then equation (15) is simplified as

$$\lambda + \gamma + N\gamma e^{-\lambda\tau} = 0. \tag{16}$$

By multiplying both sides by τ and introducing the new variables

$$\Delta = \lambda\tau, \quad A = \gamma\tau,$$

we have the following equation:

$$\Delta + NAe^{-\Delta} + A = 0. \tag{17}$$

We will next provide a complete eigenvalue analysis. Let $\Delta = \alpha + i\beta$ be a complex root. We can assume that $\beta > 0$, since if Δ is a solution, then its complex conjugate is also a solution. Then

$$\alpha + i\beta + A + NAe^{-\alpha}(\cos\beta - i\sin\beta) = 0. \tag{18}$$

The real and imaginary parts imply that

$$\alpha + A + NAe^{-\alpha}\cos\beta = 0 \tag{19}$$

and

$$\beta - NAe^{-\alpha}\sin\beta = 0. \tag{20}$$

If $\sin\beta = 0$, then $\beta = 0$, so the root is real and solves equation

$$\alpha = -A - NAe^{-\alpha}$$

implying that $\alpha < 0$ if α is a solution. If $\sin\beta \neq 0$, then

$$e^{-\alpha} = \frac{\beta}{NA\sin\beta}$$

and by substituting it into (19) we get

$$\alpha + A + NA\frac{\beta}{NA\sin\beta}\cos\beta = 0$$

or

$$\alpha = -A - \beta\cot\beta. \tag{21}$$

By substituting this relation into (20) a single-variable equation is obtained for β:

$$\beta - NAe^{A+\beta\cot\beta}\sin\beta = 0$$

or

$$\frac{\beta}{NAe^{A}} = e^{\beta\cot\beta}\sin\beta. \tag{22}$$

In order to have asymptotical stability we need conditions which guarantee that $\alpha < 0$. From (21) this is the case when the solution of equation (22) also satisfies

$$\beta\cot\beta > -A. \tag{23}$$

Let $f(\beta)$ denote the right hand side of equation (22) and

$$g(\beta) = \beta\cot\beta. \tag{24}$$

Clearly,

$$g'(\beta) = \left(\frac{\beta\cos\beta}{\sin\beta}\right)' = \frac{\sin 2\beta - 2\beta}{2\sin^2\beta} < 0$$

for all $\beta > 0$. So $g(\beta)$ strictly decreases in β. In addition,

$$\lim_{\beta \to 0+} \frac{\beta \cos \beta}{\sin \beta} = 1,$$
$$\lim_{\beta \to \frac{\pi}{2} + k\pi} \frac{\beta \cos \beta}{\sin \beta} = 0,$$
$$\lim_{\substack{\beta \to k\pi \\ (k \geq 1)}} \frac{\beta \cos \beta}{\sin \beta} = \begin{cases} -\infty & \text{from left hand side} \\ +\infty & \text{from right hand side.} \end{cases}$$

Figure 1 shows the graph of $g(\beta)$. There is a unique intercept β_k of the

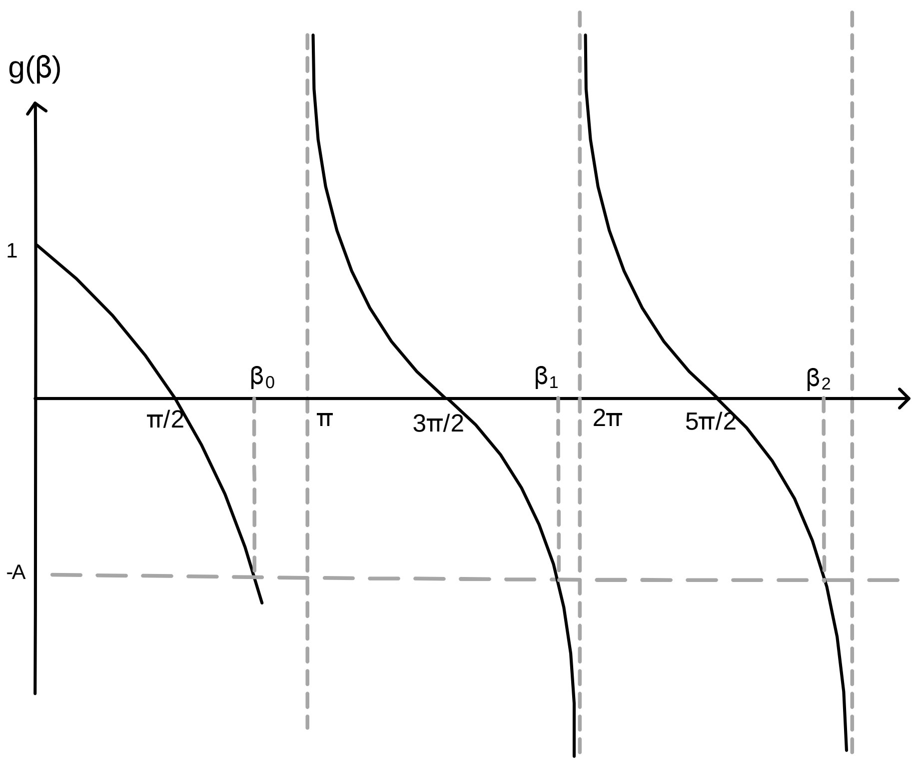

Figure 1: Graph of $g(\beta)$

horizontal line of $-A$ and the curve of $g(\beta)$ in each subinterval $(\frac{\pi}{2} + k\pi, (k+1)\pi)$, $k = 0, 1, 2, \cdots$ and $\alpha < 0$ if and only if $\beta < \beta_k$ in the subinterval. It is easy to see that

$$\lim_{\beta \to 0+} f(\beta) = 0$$

and for $k \geq 1$,

$$\lim_{\substack{\beta \to k\pi - 0 \\ (\beta \geq 1)}} f(\beta) = 0,$$

since $\beta \cot \beta \to -\infty$ as β tends to $k\pi$ from the left hand side. Similarly

$$\lim_{\substack{\beta \to k\pi + 0 \\ (k \geq 1)}} f(\beta) = \begin{cases} \infty & \text{if } k \text{ is even} \\ -\infty & \text{if } k \text{ is odd,} \end{cases}$$

and finally

$$\lim_{\substack{\beta \to \frac{\pi}{2} + k\pi \\ (k \geq 0)}} f(\beta) = \begin{cases} 1 & \text{if } k \text{ is even} \\ -1 & \text{if } k \text{ is odd.} \end{cases}$$

Simple differentiation shows that

$$\begin{aligned} f'(\beta) &= \cos\beta e^{\beta \frac{\cos\beta}{\sin\beta}} + \sin\beta e^{\beta \frac{\cos\beta}{\sin\beta}} \frac{\sin\beta\cos\beta - \beta}{\sin^2\beta} \\ &= \frac{1}{\sin\beta} e^{\frac{\beta\cos\beta}{\sin\beta}} \left(\sin 2\beta - \beta\right). \end{aligned}$$

There is a unique β^* in $(0, \frac{\pi}{2})$ such that $\sin 2\beta^* = \beta^*$, and for $\beta < \beta^*$, $\sin 2\beta > \beta$, and for $\beta > \beta^*$, $\sin 2\beta < \beta$. Therefore $f'(\beta) > 0$ if and only if either $\beta \in (0, \beta^*)$ or $\beta \in ((2k-1)\pi, 2k\pi)$, $k = 1, 2, \cdots$. Similarly $f'(\beta) < 0$ if and only if either $\beta \in (\beta^*, \frac{\pi}{2})$ or $\beta \in (2k\pi, (2k+1)\pi)$, $k = 1, 2, \cdots$. The graph of $f(\beta)$ is shown in Figure 2. The solution of equation (22) is the intercept of this graph with the linear function $\beta/(NAe^A)$. In order to have asymptotical stability we

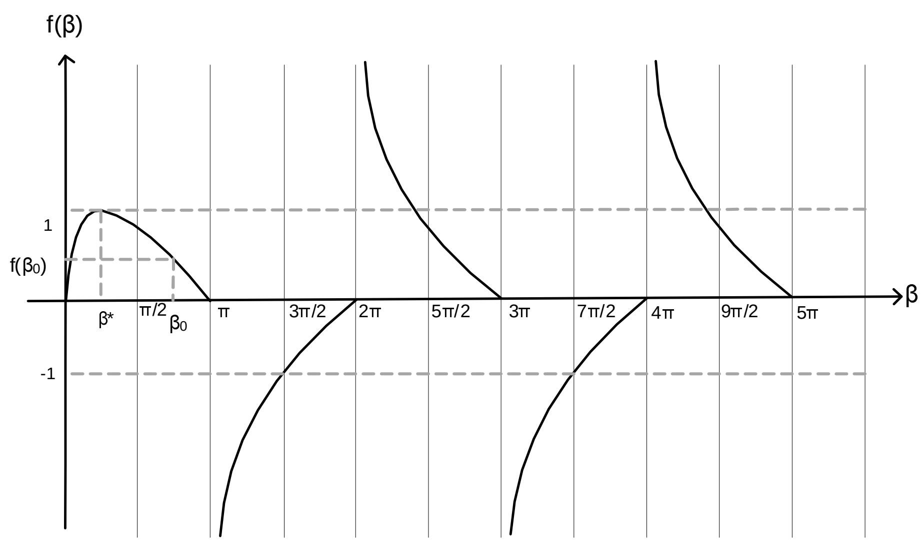

Figure 2: Graph of $f(\beta)$

have to avoid the intervals $[\beta_0,\pi]$,$[\beta_1,2\pi]$, $[\beta_2,3\pi]$, $\cdots$ etc., which happens if the slope of the linear line is greater than $f(\beta_k)/\beta_k$ for $k=0,2,4,\cdots$. That is,

$$\frac{1}{NAe^A} > \frac{f(\beta_k)}{\beta_k}$$

or

$$\begin{aligned} 1 &> \frac{NAe^A e^{\beta_k\cot\beta_k}\sin\beta_k}{\beta_k} = \frac{NAe^A e^{-A}\cos\beta_k}{\beta_k\cot\beta_k} \\ &= -N\cos\beta_k. \end{aligned} \tag{25}$$

Next we show that if $1 > -N\cos\beta_0$, then (25) holds for all $k=2,4,\cdots$. It is sufficient to show that $\cos\beta_0 \le \cos\beta_2 \le \cos\beta_4 \le \cdots < 0$ which is a simple consequence of the facts that $\beta_0 < \beta_2 < \beta_4 < \cdots$ and $\beta_k\cos\beta_k = -A$ for all k. In summary, the system is asymptotically stable if

$$1 > -N\cos\beta_0, \tag{26}$$

and unstable if

$$1 < -N\cos\beta_0,$$

since in this case the solution $\beta \in (\beta_0,\pi)$ results in positive α.

Notice that β_0 is a strictly increasing function of A, which we can denote by $h(A)$. So (26) can be rewritten as

$$1 > -N\cos(h(\gamma\tau))$$

since $A=\gamma\tau$. That is,

$$h(\gamma\tau) < \cos^{-1}\left(-\frac{1}{N}\right) \quad \left(\in\left(\frac{\pi}{2},\pi\right)\right)$$

or

$$g(h(\gamma\tau)) > g\left(\cos^{-1}\left(-\frac{1}{N}\right)\right)$$

which is the following:

$$\begin{aligned} \gamma\tau &< -\cos^{-1}\left(-\frac{1}{N}\right)\cot\left(\cos^{-1}\left(-\frac{1}{N}\right)\right) \\ &= \cos^{-1}\left(-\frac{1}{N}\right)\frac{\cos\left(\cos^{-1}\left(-\frac{1}{N}\right)\right)}{\sqrt{1-\cos^2\left(\cos^{-1}\left(-\frac{1}{N}\right)\right)}} \\ &= \frac{1}{\sqrt{N^2-1}}\cos^{-1}\left(-\frac{1}{N}\right). \end{aligned} \tag{27}$$

Let $F(N)$ denote the last expression. The stability region in the (γ,τ) plane is illustrated in Figure 3. It is interesting that it does not depend on A, only on N. Notice that $F(N)$ strictly decreases in N, $F(1)=\infty$, $F(2)=\frac{1}{\sqrt{3}}\cos^{-1}(-\frac{1}{2})=\frac{2\pi}{3\sqrt{3}}$, and $F(N)\to 0$ as $N\to\infty$.

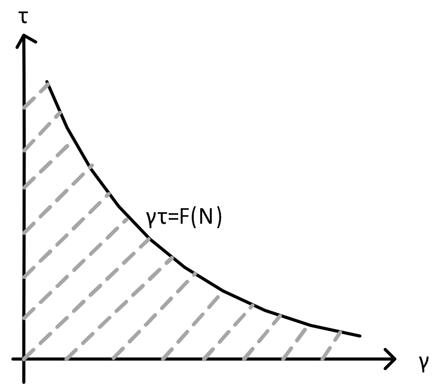

Figure 3: Stability region in the (γ, τ) plane

4 Stability switches and Hopf bifurcation

Stability switches might occur when there is a pure complex eigenvalue, $\Delta = i\beta$ with $\beta > 0$. Substituting it into the characteristic equation (17) we have

$$i\beta + A + NAe^{-i\beta} = 0$$

or

$$i\beta + A + NA(\cos\beta - i\sin\beta) = 0.$$

The real and imaginary parts imply that

$$A + NA\cos\beta = 0 \tag{28}$$

and

$$\beta - NA\sin\beta = 0. \tag{29}$$

From (28), $\cos\beta = -\frac{1}{N}$, so

$$\beta = \cos^{-1}\left(-\frac{1}{N}\right) + 2n\pi,$$

since from (29), $\sin\beta > 0$. Then (29) implies that

$$\cos^{-1}\left(-\frac{1}{N}\right) + 2n\pi - NA\sqrt{1 - \frac{1}{N^2}} = 0,$$

so

$$A_n = \frac{\cos^{-1}\left(-\frac{1}{N}\right) + 2n\pi}{\sqrt{N^2 - 1}}.$$

Notice that $A = \gamma\tau$. Consider γ fixed and select τ as the bifurcation parameter. Then $\Delta = \Delta(\tau)$, and by differentiating (17) with respect to τ yields

$$\Delta' + \gamma + N\gamma e^{-\Delta} + N\gamma\tau e^{-\Delta}(-\Delta') = 0$$

implying that

$$\Delta' = -\frac{\gamma + N\gamma e^{-\Delta}}{1 - N\gamma\tau e^{-\Delta}} = -\frac{\gamma + N\gamma \frac{-\Delta-\gamma\tau}{N\gamma\tau}}{1 - N\gamma\tau \frac{-\Delta-\gamma\tau}{N\gamma\tau}}$$
$$= \frac{\Delta}{\tau(1+\Delta+\gamma\tau)}.$$

If $\Delta = i\beta$, then

$$\Delta' = \frac{i\beta}{\tau(1+\gamma\tau+i\beta)} = \frac{i\beta(1+\gamma\tau-i\beta)}{\tau((1+\gamma\tau)^2+\beta^2)}$$

with real part

$$\operatorname{Re}\Delta' = \frac{\beta^2}{\tau((1+\gamma\tau)^2+\beta^2)} > 0$$

showing that at any critical value the real part of an eigenvalue changes sign from negative to positive. At A_0 it is a stability switch, but at A_n $(n \geq 1)$ the system is already unstable, so there is no stability switch. At $A = A_0$ Hopf bifurcation occurs giving the possibility of the birth of limit cycles. So we have the following result.

THEOREM. *If $\tau\gamma < A_0$, then the system is asymptotically stable. If $\tau\gamma > A_0$, then it is unstable. At $\tau\gamma = A_0$ Hopf bifurcation occurs.*

5 Conclusions

A special learning process in oligopolies was examined when the firms had delayed information about the market price. A complete spectrum analysis was performed. The stability region was determined and possible stability switches were found. We also verified that only the smallest threshold can lead to stability switches since later the system is unstable anyway due to the positive real part of another eigenvalue. We found that learning leads to accurate knowledge of the price function if

$$A = \gamma\tau = \frac{K}{N+1}\tau < A_0 = \frac{\cos^{-1}\left(-\frac{1}{N}\right)}{\sqrt{N^2-1}}$$

or

$$\tau < \frac{1}{K}\cos^{-1}\left(-\frac{1}{N}\right)\sqrt{\frac{N+1}{N-1}},$$

where this threshold depends on both the common speed of adjustment K and the number N of firms. Notice that it decreases in both K and N, so larger speed of adjustment and/or larger number of firms make the system less stable. In our future research we will drop the assumption that the firms have identical speed of adjustment, and will examine the spectrum based on equation (15). It will need a much more complex study.

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