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REGULAR CONTRIBUTION

A multiresolution approach for optimal defense against random attacks

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Abstract Whether it be one security expert covering more systems or reducing total man-hours, there has always been a push to do more with less. Intuitively, we realize different systems need different levels of security. To aid in this effort, we develop multiresolution attacker/defender games by combining two game theoretic approaches: resource assignment and optimal response. We use the resource assignment game to determine the level of detail necessary to build the game needed to respond optimally to attacks. To aid in the selection of a resource assignment game and an optimal response game, we present considerations and survey numerous works. Further resource savings are possible when the optimal response games share features. Even though effort sharing between systems ought to be addressed during the resource-allocation game, we present both a linear effort sharing model and a method for solving post hoc. An illustrative example demonstrates the potential savings from our technique.

Keywords Multiresolution · Games · Game theory · Attacker/defender · Security · Survey

1 Introduction

Defending assets have become one of the top priorities for network administrators and homeland defense alike. Due to the nature of the interactions between the attackers and defenders, game theory provides an appropriate framework to aid in all sorts of security breaches. Since the attacks on September 11, 2001, game theory research and application have become as popular as they were back in the cold war era. Then, as now, game theory has been studied as a mechanism for defense. However, the major difference between then and now is the nature of attackers. Today, game theory targets counter-terrorism [2,58,59] rather than international policy [9].

Defenders need to know both how to distribute resources among assets and how to protect them. While game theory has been used to provide answers to how to protect assets [47–50,63] and how to distribute resources [1,3,4,16,21,31], they provide the answers separately. However, we can do better by combining these two approaches. Thus, we propose combining these two approaches with a multiresolution approach [5,7,56,65].

Knowing that some assets require more resources to optimally defend, naturally leads to the use of multiresolution models. Methods that provide optimal response actions to attacks [49,50,63] require significant amounts of time and information to construct their models. Therefore, it is logical for the resources spent developing the attacker/defender games to be a function of the resources allocated to defending that asset. If some assets require more resources, then some models will have higher levels of detail. This is what we mean by multiresolution models.

Kobbelt et al. [36], Garland [8], Park et al. [52], and Goswami et al. [12] use multiresolution in context of computer graphics and computer vision. The basic idea is that objects (polygons, point clouds, etc.) which are further away or are partially obscured have lower levels of detail. This means they can be approximated using less information. By abstracting this approach from computer graphics and using it for decision support, threats deemed "far away" can be mod-

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eled more frugally, saving resources. While a threat being "far away" is undefined, it is clear that it depends on the probability of the attacks and the value of the assets. Therefore, we propose making an attacker/defender game's level of detail a function of the resources allocated to it.

Note that a multiresolution approach can be applied to the construction of a game or to its analysis. We will only briefly address multiresolution game analysis. In multiresolution analysis, each action A_i (a high level of abstraction) has more detailed instantiations A_{ij}. All A_{ij} simplify to A_i when the game, its consequences, or the actions' consequences are deemed sufficiently "far away". The bulk of this paper will focus on the multiresolution construction of games. By this, we mean more significant threats receive more elaborate games to minimize the potential damage.

The rest of the paper proceeds as follows. Section 2 discusses the selection of attacker/defender games. Subsections review one method for resource allocation and one method for determining optimal responses. Multiresolution games and effort sharing are covered in Sect. 3. We provide an illustrative example in Sect. 4. Conclusions and further research directions are given in Sect. 5.

2 A comparison of works

A summary and comparison of different works are in Tables 1 and 2. We only cover the works most relevant to our approach. For a larger survey, but using a different classification system, see the work by Hausken and Levitin [29]. The tables are abbreviated using the "list of title word abbreviations".[1] We also introduce more abbreviations in the text that follows.

Since we are pairing a resource-allocation (res-alloc.) game and an optimal response action game (det. resp. act.), these games should use similar frameworks. This makes the assumptions consistent. We will consider several assumptions and introduce their abbreviations. Issues to consider include whether the payoffs are stochastic (stoch.) or deterministic (non-stoch.); which moments do the stochastic frameworks optimize; what are the models; are moves simultaneous (simult.), sequential (seq.), or only one-sided (agno.); and what is the type of analysis (*i.e.,* optimization (optim.) or Nash equilibria). While consistent frameworks are not strictly required, consistency helps justify of the results.

Likewise, we prefer accurate frameworks. There are primarily two ways to improve accuracy: more detailed models and better approximations. Most of the works in Table 1 take into account system structure such as series, parallel, nearly decomposable, interdependent, or n-out-of-k systems. If the assets in question are part of a larger system, composed in any of these manners, then these works provide more detailed models. On the other hand, accuracy can be bolstered by using better approximations. According to Samuelson [57], any distribution can be characterized by its mean, variance, skewness, kurtosis, etc. Using higher moments will increase both the complexity and accuracy of the analysis. By optimizing only the expected value (EV), one assumes a linear utility function. Contrary to this, most reasonable utility functions have diminishing returns implying they are nonlinear. To account for this behavior, we must take into account higher moments. Substituting the uncertain payoffs with $\mu - \alpha\sigma^2$, a linear combination of the expected value and variance (VAR), is a common practice in the economic literature [60]. Thus, we prefer stochastic models which use higher moments. Works that use higher moments are all listed at the bottom of Table 2.

Tables 1 and 2 show a myriad of models. The works in Table 1 are built off of additive (A-), series (S-), parallel (P-), combined (C-), nearly decomposable (ND-), n-out-of-k (NooK-), interdependent or interlinked (I-) system structures. For example, Hausken [13] studies additive, series, and parallel systems (ASP-syst.). Other models include "ratio" and "difference" (diff.) contest functions [35], balls-and-bins models [55], and resource-constrained shortest-path problem (RCSPP) [34]. Many are based on Markov decision processes (MDP) and its extensions. These include multiagent Markov decision processes (MMDP), partially observable Markov decision processes (POMDP), multiagent partially observable Markov decision processes (MPOMDP), and partially observable competitive Markov decision processes (POCMDP) [33]. Some works use generic functions as models; when possible we list properties of these functions such as whether they are affine (aff.), linear (lin.), or multiplicative (mult.). However, we are unaware of any study definitively showing one model being superior. For this reason, we are indifferent toward models.

While we cannot thoroughly discuss all relevant works, we pick a few to review more in depth. The authors of Carin et al. [6] present an analysis tool called Quantitative Evaluation of Risk for Investment Efficient Strategies (QuERIES). QuERIES takes a protection map (a detailed security plan) and reverse-engineering methodologies to generate a POMDP, from which it can quantify risk and estimate probability distributions (est. prob. dist.). Hausken [18] builds on much of Hausken's earlier works to cover a large variety of different system configurations, including how to decompose some systems into a combination of serial and parallel subsystems. Levitin [39] uniquely discusses how to allocate resources (between detection and destruction) when there are unknown parameters in the ratio contest functions. Various works focus on trade-offs between defense, hiding, separation of assets, the separation strategy (sep. strat.), false targets, and redundancy.

[1] For a complete list see http://www.issn.org/services/online-services/access-to-the-ltwa/.

Table 1 Comparison & summary of related works

Related work	Objective	Payoff and moments	Model	Move type	Analysis type
Azaiez and Bier [1]	Res-alloc.	EV	Testing PS-syst.	One-sided	Optim.
Bier et al. [4]	Res-alloc.	EV	PS-syst.	One-sided	Optim.
Hausken [13]	Res-alloc.	EV	ASP-syst.	Simult.	Nash
Hausken [15]	Res-alloc.	EV	Ratio cont., CSP-syst.	Simult.	Nash
Hausken [17]	Res-alloc.	EV	Ratio cont., CISP-syst.	Simult.	Nash
Hausken [18]	Res-alloc.	EV	Ratio cont., CINDNooKSP-syst.	Simult.	Nash
Hausken [20]	Res-alloc.	EV	Ratio and diff. cont., CIPS-syst.	Simult.	Nash
Hausken and Levitin [24]	Res-alloc.	EV	Ratio cont., P-syst.	N.A.	Def. vs. sep.
Hausken and Levitin [25]	Res-alloc., sep. strat., est. prob. dist.	EV	CPS-syst.	Seq.	Nash
Hausken and Levitin [26]	Res-alloc. (def. vs. false targets)	EV	Ratio cont., S-syst.	Seq.	Nash and optim.
Hausken and Levitin [27]	Res-alloc. (def. vs. sep.)	EV	Ratio cont., P-syst.	Simult.	Nash
Hausken and Levitin [28]	Res-alloc. (2-seq. attacks)	EV	Ratio cont., P-syst.	Seq.	Nash
Hausken [14,16]	Res-alloc.	EV	Ratio cont., SP-syst.	Simult.	Nash
Hausken [21]	Res-alloc.	EV	Ratio cont., S-syst.	Seq.	Nash
Levitin [37], Levitin and Ben-Haim [41], Levitin [38,40]	Res-alloc., sep. strat., est. prob. dist.	EV	SP-syst., unmet demand	Seq.	Nash
Levitin and Hausken [42]	Res-alloc. (def. vs. redundancy)	EV	Ratio cont., P-syst., unmet demand	Seq.	Nash
Levitin and Hausken [43]	Res-alloc., sep. strat., est. prob. dist.	EV	Ratio cont., P-syst., unmet demand	Seq.	Nash
Levitin and Hausken [45]	Res-alloc. (def. vs. sep.)	EV	Ratio cont., SP-syst.	Seq.	Nash
Peng et al. [53]	Res-alloc. (def. vs. false targets)	EV	Ratio cont., SP-syst.	Seq.	Nash and optim.

Multiple methods from Tables 1 and 2 form good combinations to determine multiresolution optimal responses. We will review two which work well together. For the rest of this paper, we will be using the resource allocation from Szidarovszky and Luo [62] to calculate the level of detail. This level of detail will be used to build general purpose games of the type mentioned in [63]. We chose these two because they both account for variance-based risk, both can produce optimal solutions taking the opponent's prior probabilities into account, Valenzuela et al. [63] allow games to reuse elements from other games, and Valenzuela et al. [63] provide a model with nonlinear rules. These two are reviewed in Sects. 2.1 and 2.2.

2.1 The resource assignment game

Szidarovszky and Luo [62] suppose I independent potential targets need defending. Let a particular target be represented by $i \in \{1, 2, \ldots, I\}$. Each target is valued as being worth v_i. The attacker's efforts, n_i, are known and the defenders efforts, m_i, are to be determined. Using the standard ratio contest function and the target's value, the defender avoids $\frac{v_i\, m_i}{m_i+n_i}$ damage at a cost of $c_i\, m_i$, where c_i is the unit cost to defend. This cost is only incurred if the target is attacked. Each target has the probability p_i of being attacked. Therefore, the defender's payoff is

$$u_i = \begin{cases} \frac{v_i\, m_i}{m_i+n_i} - c_i\, m_i & \text{with probability } p_i \\ 0 & \text{with probability } 1 - p_i \end{cases} \tag{1}$$

The objective function is the certainty equivalent [60] of this random payoff:

$$D = E\,[u] - r\, Var\,[u]$$

$$\text{s.t.} \quad B \geqq \sum_{i=1}^{I} c_i\, m_i$$

$$u = \sum_{i=1}^{I} u_i$$

Table 2 Comparison & summary of related works

Related work	Objective	Payoff and moments	Model	Move type	Analysis type
Bier et al. [3]	Res-alloc. (two locations)	EV	Gen.	Simult. and seq.	Nash
Carin et al. [6]	Quant. risk, est. prob. dist.	EV	POMDP	Agno.	Agno.
Golany et al. [10]	Res-alloc. (schedule)	Non-stoch.	RCSPP	N.A.	Optim.
Golany et al. [11]	Res-alloc. (schedule)	EV	Damage while wait.	Simult.	Nash and optim.
Hausken et al. [23]	Res-alloc. (vs. terror. and nature)	EV	Ratio cont., aff.	Simult. and seq.	Nash and optim.
Hausken and Bier [22]	Res-alloc. (1 asset)	EV	Ratio cont.	Simult.	Nash (n-player)
Hausken [19]	Res-alloc. (2 assets)	EV	Ratio cont.	Simult.	Nash
Hausken and Zhuang [30]	Res-alloc. (2-seq. attacks)	EV	Ratio cont., aff.	Seq.	Nash
Hausken and Zhuang [31]	Res-alloc. (T-period defense/attack)	EV	Ratio cont.	Seq.	Nash
Hausken and Zhuang [32]	Res-alloc. (1 asset, myopic repeated)	EV	Ratio cont., aff.	Seq.	Nash
Levitin [39]	Res-alloc. (def. vs. hiding)	EV and worst-case	Ratio cont., mult.	Simult.	Nash
Levitin and Hausken [44]	Res-alloc. (2-seq. attacks)	EV	Ratio cont.	Seq.	Nash
Shen et al. [61]	Det. resp. act. (many players with teams)	EV	MMDP w/ hierarchical payoffs	Simult.	Nash
Wang et al. [64]	Max. prob. of service availab.	Stoch.	Balls-and-bins Poisson	Agno.	Optim.
Zhang and Ho [66]	Ident. pivots of attack schemes	EV	MPO-MDP	Simult.	Optim.
Zhuang and Bier [67]	Res-alloc.	EV	Gen.	Simult. and seq.	Nash
Zhuang et al. [68]	Res-alloc. and signaling	EV	Ratio cont.	Partially simult.	Nash
Zonouz et al. [69]	Det. counter-measures	EV	POCMDP	Seq.	Nash
Luo et al. [48]	Det. multistage block levels	EV	Custom	Seq.	Nash
Luo et al. [49]	Det. resp. act.	EV, VAR	Gen.	Seq.	Optim.
Luo et al. [50]	Det. resp. act.	EV, VAR	Gen.	Seq.	Optim.
Szidarovszky and Luo [62]	Res-alloc.	EV, VAR	Ratio cont.	Agno.	Optim.
Valenzuela et al. [63]	Det. resp. act.	EV, VAR	Aff., state space	Seq.	Nash and optim.

where $r \in \mathbb{R}$ is the risk attitude of the decision maker, $E[u]$ is the expected value, $Var[u]$ is the variance of u, and B is the defender's budget.

This optimization problem is non-convex, the solution of which has numerical difficulties, and the computed result has no guarantee it is globally optimal. By introducing new decision variables, the objective function D transforms into a concave function. Since the constraints also become concave, the problem comes a standard concave programming problem. It can be solved with standard gradient algorithms [46]. Moreover, the problem further simplifies if the budget is sufficiently large. Then the problem reduces to the following three-stage problem. First solve for m_i^*, the stationary point for (1) and calculate the corresponding payoff u_i^*. For all $m_i^* \leq 0$, use $m_i = 0$ to calculate u_i^*. Second, solve for u_i and an auxiliary variable[2] Q, such that

$$\sum_{i=1}^{I} p_i \, u_i(Q) = Q \tag{2}$$

where

$$u_i(Q) = \begin{cases} Q + \frac{1}{2r} & \text{if } \; Q \leq u_i^* - \frac{1}{2r} \\ u_i^* & \text{otherwise} \end{cases} \tag{3}$$

This can be solved easily by finding the intersection of a 45 degree line (Q) passing through the origin with the piecewise linear function $\sum_{i=1}^{I} p_i u_i(Q)$. The solution is unique provided r is finite. Then given u_i solve for m_i according

[2] Q allows us to solve for u_i despite each u_i depending on all other u_j.

to (1). If two solutions exist, take the smaller positive solution.

While the resource allocation is useful in itself, it does not prescribe a course of action to defend the target. However, the resources allocated to each target can then be used to determine how much effort (*e.g.,* man-hours) should be used to determine such courses of action. Since game theory can also be used to prescribe courses of action, the resources distributed to a target ought to, in part, be used to construct these elaborate games.

2.2 The optimal response framework

Valenzuela et al. [63] introduce both a game theoretic framework and a game builder tool. The framework is similar to other repeated two-person games with the notable exceptions that it includes:

1. a rule set (*i.e.,* when the strategies are available),
2. a state space that actions (strategies) influence,
3. initial conditions for each player,
4. a linear mapping from the state space to the payoff for each player,
5. risk attitudes for each player, and
6. optional prior probabilities.

Nevertheless, the most important detail about this work is that it possesses a system to reuse actions and rule sets. This feature makes this work particularly attractive for building multiresolution games when reuse of rules or actions is possible.

Unlike many repeated games which have been studied in the literature, the available actions change depending on the triggered rules. Each rule can be mathematically defined as a seven-tuple consisting of:

1. a set of triggering players,
2. a set of triggering actions,
3. an add/remove operator,
4. a set of add/remove actions,
5. a set of affected players,
6. the rule's life span in number of turns, and
7. a Boolean describing if it is initially active.

A rule is triggered when any of the triggering actions are performed by a triggering player. When a rule is triggered, it will allow/disallow strategies for one or both players for the specified number of turns. This allows for complex interactions. For instance, after a player has been attacked first, an action "attack" might be replaced with the action "retaliate", where "retaliate" effects the state of the game differently from the "attack".

To provide better introspection into the state of the game, Valenzuela et al. [63] use a state space-based cost-benefit model. A state space-based cost-benefit model is a collection of real numbers describing the resources or factors for each player. These resources/factors provide meaningful information about the state of the game. They may come from popular analyzes such as PMESII (political, military, economic, social, infrastructure, and information systems)[3] or ASCOPE (areas, structures, capabilities, organizations, people, events) [51]. A state space allows for a more natural representation of the impact of actions and provides insight into why an outcome would be considered good or bad.

The state space-based cost-benefit model provides the fundamental way strategies interact and provides the basis for the player payoffs. Strategies perform affine transformations on the state space. Let s be the old state concatenated with the value 1.0 and M be the affine transformation. Then the new state is $s' = Ms$. For example, an action operating under the ASCOPE model will have a 12×13 matrix, and the state space will be a vector of length 12. Likewise, each player has a linear transformation of the state vector which generates their payoff. Again assuming the ASCOPE model, each player would have 12 weights describing the relative importance of each component of the state.

Each player also has a real number describing their risk attitude r. Larger values of r mean the player wants to avoid risk (have tighter guarantees), and smaller values of r mean the player is risk seeking. This risk attitude is interpreted identically to the risk attitude in Szidarovszky and Luo [62]. Let an action a have the outcome distribution u_a. Then the objective function maximized is

$$E[u_a] - r\, Var[u_a]. \tag{4}$$

The variables and methods in Algorithm 1 are described as follows. The variable $players$ contains an attacker and defender player. Each player contains their risk attitudes, initial state vector, and a linear payoff function as previously defined. The variable $actions$ is a set of affine transition matrices. $rules$ is a set of rules, with each rule containing the aforementioned seven-tuple. $maxDepth$ is an integer which limits the depth the game tree is expanded. The method BuildProbabilisticGameTree builds the game tree out to $maxDepth$ and works from the leaves back to the root assigning probabilities according to the details in the work of Valenzuela et al. [63]. GetValidActions returns the available actions at the root node, and StatsFromGameTree collects the variance and expected value for each action at the root of the game tree.

The game proceeds roughly as shown in Algorithm 1. Some steps were hidden for brevity, specifically the steps for generating the attacker's action distribution and exploring the game tree. Using the defender's prior action distribution, the attacker calculates the payoff distribution for each of his/her

[3] See http://pmesii.dm2research.com/wiki/index.php/Main_Page.

Algorithm 1 The game proceeds as follows

```
Procedure ANALYSIS(players, actions, rules, maxDepth)
turn ← 0
history ← empty
gameTree ← empty
while turn < enough do
  if ISEVEN(turn) then                      ▹ Even turn → attacker's turn
    risk ← player.attacker.risk
  else
    risk ← player.defender.risk
  end if
  scores ← empty
  statistics ← empty
  lowCut ← −∞                               ▹ Lower bound on performance
  gameTree ← BUILDPROBABILISTICGAMETREE(
      gameTree, players, actions, rules,
      turn, history, maxDepth − 1)
  validActions ← GETVALIDACTIONS(gameTree, turn)
  for all a ∈ validActions do               ▹ Collect statistics
    actStats ← STATSFROMGAMETREE(
        players, a, gameTree)
    lowCut ← MAX(lowCut, actStats.min)
    Append actStats → statistics
  end for
  for all actStats ∈ statistics do
    if actStats.max < lowCut then           ▹ Reject dominated slns
      Append − ∞ → scores
    else                                    ▹ Calculate utility
      Append actStats.eV − actStats.var · risk → scores
    end if
  end for
  bestValue ← −∞
  bestA ← −1
  for i = 0 thru validActions.length do     ▹ Find the best action
    if bestValue < scores[i] then
      bestValue ← scores[i]
      bestA ← i
    end if
  end for                                   ▹ Advance the game one step
  Append action[bestA] → history
  turn ← turn + 1
end while
```

actions. Expression 4 converts the payoff distribution from each action into a certainty equivalent. The certainty equivalents are then mapped onto probabilities using an additive contest function [35]. The defender then determines the optimal defense with respect to the attacker's distribution. This follows the same process as the attacker's process.[4] For more details can be found in [63].

3 Multiresolution and shared effort

Clearly, creating games to determine *how* to respond to an attack is time and resource consuming. When modeling multiple heterogeneous targets, given a limited budget, multiresolution modeling is a natural fit. Hence, we propose using the resource allocation of Szidarovszky and Luo [62] and distributing a portion of those resources to the construction of games in the framework from Valenzuela et al. [63].[5] Combining these two approaches, not only determines how resources ought to be divided, but provides the first step in constructing games of optimal fidelity given limited resources.

Just to be clear, we will informally define a game's resolution (also referred to as its level of detail). When constructing a game, one must consider the level of abstraction for the: available strategies, corresponding rules, initial states, payoffs, risk attitude, prior probabilities, and effects of actions.[6] In general, we say a game's resolution increases as the number of actions, number of rules, and their corresponding level of detail increases. Also, the more accurate the estimates of the game's initial conditions, players' payoff functions, and risk attitude, the higher the resolution of the game. Thus, in our context, multiresolution games simply mean that some games will be more elaborate than others.

Now suppose an assignment game has already assigned the resources for each target. If anticipated attacks on these targets lack similarities, all that is left is to build the games. However, similarities between targets or attackers allow effort spent developing one game to be applied to similar games. Then one must address the issue of shared effort.

3.1 Shared effort

It is most correct to push the issue of effort sharing back to the resource-allocation problem, as nonlinearities and constraints from the original problem are lost when solving the issue of optimal resource sharing post hoc. For example, many objective functions model diminishing returns. If effort sharing can be reasonably integrated into the resource-allocation problem, then optimal effort sharing can account for these diminishing returns. However, combining the effort sharing problem with the resource-allocation problem may be unreasonable. The work to combine the two may be infeasible or make the analysis more complex.

When the effort sharing problem cannot be combined with the resource-allocation problem, the problem can be solved post hoc, provided some simple assumptions. (1) If

[4] An ε-Nash equilibrium occurs when the attacker's assumed prior distribution of the defender's responses are sufficiently close to the defender's actual choice of defense actions.

[5] Again any pair can used together, but the more similar the frameworks between the resource-allocation problem and the optimal response problem, the more consistent the assumptions.

[6] Further discussion of how resources ought to be spent on each component of the game is left open for future research.

the resource-allocation problem is sufficiently linear around its solution and (2) the fraction of resources, ε, dedicated to constructing attacker/defender games is sufficiently small, then a linear effort sharing problem can provide a near optimal solution post hoc. As such, the rest of this section proposes and solves a linear resource sharing problem.

The problem is to find w_i, the work (in terms of resources *with* resource sharing) dedicated to building the attacker/defender game for asset i. Let m_i be the desired quantity of resources assigned (without resource sharing) to defend target i, $\underline{\underline{A}}$ be the matrix describing how effort may be shared between different projects, and A_{ij} an element of $\underline{\underline{A}}$ is the percentage of work that transfers from asset j to asset i. Typically $\underline{\underline{A}}$ is symmetric with $A_{ii} = 1.0$ and all elements nonnegative.[7] Without any constraints, the problem is

$$\underline{\underline{A}} \cdot \underline{w} = \underline{m} \cdot \varepsilon. \tag{5}$$

We notice that this equation is similar to the usual linear input-output models in economic theory.

Depending on the nature of $\underline{\underline{A}}$, no solution or multiple solutions to (5) may exist. These problems are routinely solved by using either a pseudoinverse or linear programming. The pseudoinverse of $\underline{\underline{A}}$ solves both of these problems by providing a least squares fit [54]. However, we have four reasons why we wish to avoid it.

1. The pseudoinverse always penalizes extra effort, even when the effort is "free" due to resource sharing.[8]
2. When multiple solutions exist, the pseudoinverse does not necessarily provide the solution that minimizes the total work $\sum_{i=1}^{I} w_i$.
3. The pseudoinverse may produce solutions with negative work, $w_i < 0$.
4. The pseudoinverse does not extend well to account for further constraints.

This leads us to our second approach to resolving the problem when (5) has no or multiple solutions. The issues with using the pseudoinverse can be fixed by switching to quadratic or linear programming. If no exact solution exists, we want an approximate solution. Quadratic and linear programming can minimize either one-sided (too little effort) or two-sided (too much or too little effort) error, $\underline{e}$. Use quadratic programming to minimize $\sum_{i=1}^{I} e_i^2$ and linear programming to minimize $\sum_{i=1}^{I} e_i$. When multiple solutions exist, we want to minimize $\sum_{i=1}^{I} w_i$. Thus, we arrive at a two-objective program, where the first objective is to minimize the error and the second objective is to minimize the cost.

For simplicity, we will use a lexicographic[9] multiobjective linear programming framework:

$$\begin{aligned} &\min_{\underline{w},\underline{e}} \left(\sum_{i=1}^{I} e_i, \sum_{i=1}^{I} w_i \right) \\ &\text{s.t.} \quad e_i \geq \left(m_i \cdot \varepsilon - \sum_{j=1}^{I} A_{ij} \cdot w_j \right) \quad \forall i \\ &\qquad\; e_i \geq 0 \qquad\qquad\qquad\qquad\quad \forall i \\ &\qquad\; w_i \geq 0 \qquad\qquad\qquad\qquad\quad \forall i \end{aligned} \tag{6}$$

This resolves the four issues with the pseudoinverse. e_i represents the value from a hinge-loss function, only taking a positive value when a game lacks level of detail.[10] This avoids penalizing too much effort when it is free due to resource sharing. Because $\sum_{i=1}^{I} w_i$ is explicitly minimized, a solution will provide the least overall effort to minimize deficiencies. The linear constraints force the work to be nonnegative, and this linear program can easily accommodate more constraints.

We consider two additional sets of constraints to help make the post hoc resource sharing problem more realistic. First, two games may share only a limited set of features. Suppose that at most $\widetilde{w_{ij}}$ effort from project j applies to project i, with the maximum being T_{ij}.

$$\widetilde{w_{ij}} = \min \left\{ w_j, T_{ij} \right\} \quad \forall i, \forall j : i \neq j$$

Moreover, if we drop the condition that $i \neq j$, this set of constraints can also account for a maximum budget for developing a single project.

The second constraint models a limited set of common features, which once completed, leave only a unique work for preparing the game. For example, if games only share the attacker, resource sharing halts once the attacker's objectives, possible strategies, and assumptions are approximated. To model this, we limit the total amount of shared work a game may receive, regardless the source. Thereupon, let project i receive no more than s_i total shared work. Mathematically, this condition can be given as

[7] These are only typical conditions and not requirements.

[8] When $\underline{\underline{A}}$ has no negative entries, the only way to compensate for an asset getting too much effort would be to give another project insufficient effort.

[9] By lexicographic we mean the first objective is minimized, then the second objective is minimized with the additional constraint that the first objective remains at its optimal value.

[10] To minimize the square of the deficiencies, use a quadratic program to minimize $e^{\intercal} e$.

$$sw_i = \min\left\{\sum_{j\neq i} A_{ij}\cdot\widetilde{w_{ij}},\ s_i\right\}\quad\forall i$$

If $s_i \geq \sum_{j\neq i} A_{ij}T_{ij}$ the constraint sw_i is redundant and can be eliminated prior to solving.

With constraints, the most general problem becomes

$$\begin{array}{lll}
\min\limits_{\underline{w},\underline{e},\underline{\alpha},\underline{sw},\underline{\underline{\widetilde{w}}}} & \left(\sum_{i=1}^{I} e_i,\ \sum_{i=1}^{I} w_i\right) & \\
\text{s.t.} & \alpha_i = A_{ii}\cdot\widetilde{w_{ii}} + sw_i & \forall i\\
& e_i \geq (m_i\cdot\varepsilon - \alpha_i) & \forall i\\
& e_i \geq 0 & \forall i\\
& w_i \geq 0 & \forall i\\
& sw_i \leq \sum_{j\neq i} A_{ij}\cdot\widetilde{w_{ij}} & \forall i\\
& sw_i \leq s_i & \forall i\\
& sw_i \geq 0 & \forall i\\
& \widetilde{w_{ij}} \leq w_j & \forall i,\forall j\\
& \widetilde{w_{ij}} \leq T_{ij} & \forall i,\forall j\\
& \widetilde{w_{ij}} \geq 0 & \forall i,\forall j,
\end{array}\tag{7}$$

where α_i is the sum of the total shared work and individual work, the following three constraints are from (6), and the last six constraints define the maximum amount of shared work. This is a multiobjective linear program which can be solved efficiently with standard methods [46]. Again, this is the most general problem, which can usually be simplified given certain problem specifics.

3.2 Special cases

We will now consider assumptions to simplify the problem and obtain special cases. First, assume $A_{ij} \geq 0\ \forall i, \forall j$, meaning that effort spent developing one attacker/defender game does not harm the development of another game. Additionally, assume $A_{ii} > 0\ \forall i$; this means that work on the ith attacker/defender game makes progress toward completing that game. Also assume that each project receives a satisfactory budget ($\widetilde{w_{ii}} \leq T_{ii}$).

These assumptions allow us to complete the first step of the lexicographic multiobjective linear program symbolically. Namely, we know $\sum_{i=1}^{I} e_i = 0$, which allows us to remove it from the objectives and introduce it as a constraint. To see this note that $\sum_{i=1}^{I} e_i$ achieves its minimum at zero when each $e_i = 0$. To set $e_i = 0$, it is necessary to have a sufficiently large α_i. Since $A_{ii} > 0$, a sufficiently large $\widetilde{w_{ii}}$, makes α_i sufficiently large. From the assumption of a satisfactory budget, $\widetilde{w_{ii}}$ is only bounded from above by w_i. Thus, $e_i = 0, \forall i$ by setting $\widetilde{w_{ii}} = w_i$ and making w_i sufficiently large. This reduces the optimization problem to the following single objective linear program:

$$\begin{array}{lll}
\min\limits_{\underline{w},\underline{\alpha},\underline{sw},\underline{\underline{\widetilde{w}}}} & \left(\sum_{i=1}^{I} w_i\right) & \\
\text{s.t.} & \alpha_i = A_{ii}\cdot\widetilde{w_{ii}} + sw_i & \forall i\\
& 0 \geq (m_i\cdot\varepsilon - \alpha_i) & \forall i\\
& w_i \geq 0 & \forall i\\
& sw_i \leq \sum_{j\neq i} A_{ij}\cdot\widetilde{w_{ij}} & \forall i\\
& sw_i \leq s_i & \forall i\\
& sw_i \geq 0 & \forall i\\
& \widetilde{w_{ij}} \leq w_j & \forall i,\forall j\\
& \widetilde{w_{ij}} \leq T_{ij} & \forall i,\forall j : i\neq j\\
& \widetilde{w_{ij}} \geq 0 & \forall i,\forall j,
\end{array}\tag{8}$$

We will address four relaxations which makes the optimization problem (7) or (8) easier to solve including attacker/defender games having: perfectly unique, overlap without constraints, overlap with limits on only the total shared effort, overlap with constraints on only shared work between related games. When games are perfectly unique, all the assets are sufficiently distinct such that effort spent building an attacker/defender game cannot be reused for another asset's game. Thus, $\underline{\underline{A}}$ is simply the identity matrix. The effort vector is simply a scaled version of assigned resources:

$$\underline{w} = \underline{m}\cdot\varepsilon.\tag{9}$$

Overlap without constraints mean the assets share identical features. As features are improved, it can be used elsewhere without limits. This means (6) must be solved.

Suppose each game has unique features. These features cannot come from other games. While effort sharing limitations between any two games may be negligible, effort sharing across all games is limited. When limits are only imposed on total shared effort the linear program (7) simplifies to

$$\begin{array}{lll}
\min\limits_{\underline{w},\underline{e},\underline{\alpha},\underline{sw}} & \left(\sum_{i=1}^{I} e_i,\ \sum_{i=1}^{I} w_i\right) & \\
\text{s.t.} & \alpha_i = A_{ii}\cdot w_i + sw_i & \forall i\\
& e_i \geq (m_i\cdot\varepsilon - \alpha_i) & \forall i\\
& e_i \geq 0 & \forall i\\
& w_i \geq 0 & \forall i\\
& sw_i \leq \sum_{j\neq i} A_{ij}\cdot w_i & \forall i\\
& sw_i \leq s_i & \forall i\\
& sw_i \geq 0 & \forall i.
\end{array}\tag{10}$$

Again if the above mentioned three assumptions ($A_{ij} \geq 0$, $A_{ii} > 0$, and $\widetilde{w_{ii}} \leq T_{ii}\ \forall i, \forall j$) are met, all e_i are constrained to zero and the problem becomes a single objective linear program.

Let us consider the scenario where each game shares only a small overlap with the other games, but the games are not particularly unique. In other words, it may be possible to entirely build a game out of components from other games. Hence, the effort shared between any two games is limited. This also implies a limit on the sum of shared efforts and subsumes the previous case by allowing $s_i = \sum_{j \neq i} A_{ij} T_{ij}$. When limits are imposed on only effort sharing between games, the linear program (7) simplifies to

$$\begin{array}{lll} \min\limits_{\underline{w},\underline{e},\underline{\alpha},\underline{sw},\underline{\underline{\widetilde{w}}}} & \left(\sum\limits_{i=1}^{I} e_i, \ \sum\limits_{i=1}^{I} w_i \right) & \\ \text{s.t.} & \alpha_i = \sum\limits_{j=1}^{I} A_{ij} \cdot \widetilde{w_{ij}} \quad \forall i & \\ & e_i \geq (m_i \cdot \varepsilon - \alpha_i) \quad \forall i & (11) \\ & e_i \geq 0 \quad \forall i & \\ & w_i \geq 0 \quad \forall i & \\ & \widetilde{w_{ij}} \leq w_j \quad \forall i, \forall j & \\ & \widetilde{w_{ij}} \leq T_{ij} \quad \forall i, \forall j & \\ & \widetilde{w_{ij}} \geq 0 \quad \forall i, \forall j. & \end{array}$$

Similarly, if the three aforementioned assumptions are true, then all e_i are constrained to zero and the problem becomes a single objective linear program.

4 Illustrative example

We will extend the example given in [62]. Given:

- three assets of equal value $v_1 = v_2 = v_3 = 4$,
- the attacker engages any target with equal effort $n_1 = n_2 = n_3 = 1$,
- the costs to defend each asset are $c_1 = c_2 = 1$, $c_3 = 4$,
- the probability of attack for each asset is the same $p_1 = p_2 = p_3 = \frac{1}{3}$, and
- the risk attitude is $r = 1$.

Then:

- the stationary points are $u_1^* = u_2^* = (\sqrt{4} - \sqrt{1})^2$ and $u_3^* = 0$,
- according to (2) and (3), $u_1 = u_2 = 1$ and $u_3 = 0$, and
- the corresponding optimal resource distribution is $m_1 = m_2 = 1$ and $m_3 = 0$.

This is the solution to the resource-allocation problem. More detail can be found in [62].

We will now solve the effort sharing post hoc. Assume:

- $\underline{\underline{A}} = \begin{bmatrix} 1 & 0.25 & 0.5 \\ 0.25 & 1 & 0.25 \\ 0.5 & 0.25 & 1 \end{bmatrix}$,
- $\underline{\underline{T}} = \begin{bmatrix} 1 & 0.2 & 0.4 \\ 0.2 & 1 & 0.2 \\ 0.4 & 0.2 & 0 \end{bmatrix}$,
- $\underline{s} = [1, 1, 1]$, and
- $\varepsilon = 1$.

We can solve the problem by using the special form (11) as all s_i are redundant since $s_i > \sum_{j \neq i} A_{ij} T_{ij}$:

- $s_1 = 1 > \underbrace{0.2}_{T_{12}} \underbrace{0.25}_{A_{12}} + \underbrace{0.4}_{T_{13}} \underbrace{0.5}_{A_{13}} = 0.25$,
- $s_2 = 1 > \underbrace{0.2}_{T_{21}} \underbrace{0.25}_{A_{21}} + \underbrace{0.2}_{T_{23}} \underbrace{0.25}_{A_{23}} = 0.10$, and
- $s_3 = 1 > \underbrace{0.4}_{T_{31}} \underbrace{0.5}_{A_{31}} + \underbrace{0.2}_{T_{32}} \underbrace{0.25}_{A_{32}} = 0.25$.

Furthermore, $T_{ii} A_{ii} \geq m_i \, \forall i$ and $A_{ij} > 0 \, \forall i, j$ imply a sufficient budget, so all $e_i = 0 \forall i$. This means we only have to minimize the work done, as all m_i can be met or exceeded:

$$\begin{array}{lll} \min\limits_{\underline{w},\underline{\alpha},\underline{\underline{\widetilde{w}}}} & \left(\sum\limits_{i=1}^{I} w_i \right) & \\ & \alpha_i = \sum\limits_{j=1}^{I} A_{ij} \cdot \widetilde{w_{ij}} \quad \forall i & \\ & 0 \geq (m_i \cdot \varepsilon - \alpha_i) \quad \forall i & (12) \\ & w_i \geq 0 \quad \forall i & \\ & \widetilde{w_{ij}} \leq w_j \quad \forall i, \forall j & \\ & \widetilde{w_{ij}} \leq T_{ij} \quad \forall i, \forall j & \\ & \widetilde{w_{ij}} \geq 0 \quad \forall i, \forall j. & \end{array}$$

A standard linear program solver provides the following answer:

- $\sum\limits_{i=1}^{3} w_i = 1.9$
- $\underline{w} = [0.95, 0.95, 0.00]$,
- $\underline{\alpha} = [1.00, 1.00, 0.00]$, and
- $\underline{\underline{\widetilde{w}}} = \begin{bmatrix} 0.95 & 0.20 & 0.00 \\ 0.20 & 0.95 & 0.00 \\ 0.00 & 0.00 & 0.00 \end{bmatrix}$.

Thus, resource sharing reduces the total effort (and hence resources) by 5 %.

The multiresolution principle now states that deeper analysis should be conducted according to the resources assigned. Now one begins the task of constructing the games to determine the optimal actions to take for defending assets i_1 and i_2. Specifically, these extended games require the information as specified in Sect. 2.2, including the actions, rules, initial state, linear mappings from the state space to the payoffs, and optional prior probabilities.[11] The time spent developing these optimal response games should be proportional

[11] The player risk attitude should be inherited from the resource-allocation game.

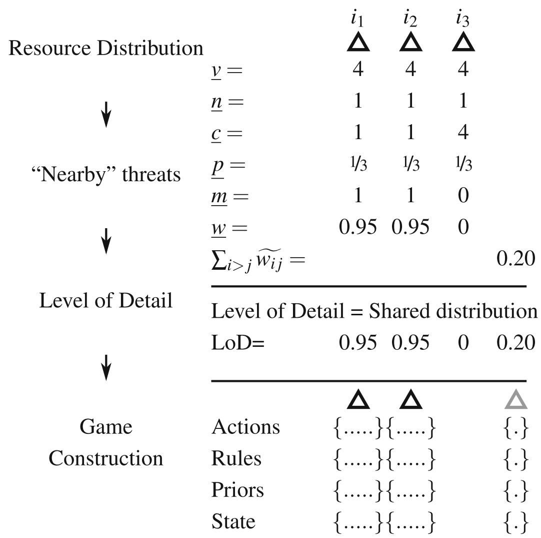

Fig. 1 The resource distribution is used to determine which threats are "nearby" to determine the desired level of detailed needed to construct or analyze a game. Resource i_3, is so "far away" that no effort should be spent on its game

to the resources assigned above. i_3 being deemed too "far away" to build its optimal response game is an example of how the multiresolution principle saves work.

The whole process is illustrated in Fig. 1. First the resource assignment game is analyzed to determine the resource distribution before resource sharing, $\underline{m}$. Solving the post hoc effort sharing problem produces $\underline{w}$, the resource distribution after work is reused. The gray triangle in Fig. 1 represents the shared work which need only be constructed once and integrated into the games for i_1 and i_2. Then the optimal response games are constructed according to the budget.

5 Conclusions

Game theory has been instrumental for national security and dealing with security breaches. On approach for handling security is the use of resource-allocation games. Given multiple assets, how much effort should go into protecting each asset. Another approach has been to build attacker/defender games to optimize responses to attacks.

Nevertheless, building optimal response games is time-consuming. This problem is exacerbated when protecting multiple assets. Since, protecting some assets more than others is worthwhile, some attacker/defender games should be built to higher levels of detail. However, we ameliorate the expense of building multiple attacker/defender games by constructing multiresolution games. Our multiresolution approach combines both the resource-allocation game and the attacker/defender game. It allows lower priority assets to use games constructed at a lower level of detail (*i.e.,* with fewer and more generic actions). The level of detail is determined by a resource-allocation game.

We also reviewed a large number of resource-allocation games and attacker/defender games. The resource-allocation game and attacker/defender games ought to be selected such that they share similar assumptions and similar models. For instance, the resource-allocation and optimal response games we used share the same stochastic framework and made use of a second-order approximation of the utility function.

We introduce a bi-objective linear effort sharing model. This model describes the fraction of effort shared, the limitations of effort sharing between any two attacker/defender games, and the limit that one attacker/defender game may benefit in general. With a few reasonable assumptions, the model can be simplified to a single objective linear program. The model suggests that when no effort sharing is possible, the efforts ought to be proportional to the resources allocated to the individual targets. For optimality, effort sharing ought to be incorporated into the resource-allocation problem whenever feasible. Nevertheless, when this is impossible or unnecessary, an approximate solution can be solved for post hoc.

In solving the multiobjective optimization problems, we selected the lexicographic method. However, a large variety of solution concepts and methods are drawn from the literature. It is an interesting research problem to compare the application of the different methods and the obtained solutions.

Further research could be conducted to determine the effects of mis-estimating games. Game theory is missing a fundamental tool that is present in other fields. Information theory has the Kullback-Leibler divergence, statistics has confidence intervals, and chaos theory has the Lyapunov exponent. It would be useful, if even possible, to extend some sort of sensitivity analysis or error measure to the creation of games. Relevant questions include "what are the consequences of omitting/adding strategies", "how accurately should payoff functions be modeled", "how abstract/detailed should strategies be", and "what is the ideal effort to spend on refining the accuracy of strategies?" If possible, such an extension could be used to determine how to optimally distribute resources when constructing a game.

Multiresolution analysis should be further studied as it may improve the game's fidelity. If uncertainties accumulate in the analysis of repeated games, using more detailed actions near the root of the game tree may provide a benefit. Switching to the less detailed actions further from the root node would compensate for the initial increase in the game's fan out. Alternatively, multiresolution analysis may provide techniques to control a game's fan-out to best suit a machine's resources.

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